

## On the second reference state and complete eigenstates of the open XXZ chain

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ABSTRACT: The second reference state of the open XXZ spin chain with non-diagonal boundary terms is studied. The associated Bethe states exactly yield the second set of eigenvalues proposed recently by functional Bethe Ansatz. In the quasi-classical limit, two sets of Bethe states give the complete eigenstates of the associated Gaudin model.

KEYWORDS: Bethe Ansatz, Lattice Integrable Models.

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## 1. Introduction

The open XXZ quantum spin chain has played a fundamental role in the study of quantum integrable systems with various boundary interactions, which appeared in statistical mechanics, condensed matter and quantum field theory. Although the special case of diagonal boundary terms was solved long ago [1–3], Bethe Ansatz solutions for non-diagonal boundary terms where the boundary parameters obey some constraints have been proposed only recently by various approaches [4–9]. It is found that in order to obtain the complete spectrum of the model two sets of Bethe Ansatz equations and consequently two sets of eigenvalues are *needed* [10, 9], in contrast with the diagonal boundary case [3]. This suggests that in the framework of algebraic Bethe Ansatz there should exist two reference states (or pseudo-vacuum states) corresponding to the two sets of Bethe Ansatz equations and eigenvalues. However, to our knowledge only one reference has been constructed so far [5, 6]. Moreover, the explicit expressions of the complete reference states and associated Bethe states are of great *importance* for investigating correlation functions of the model [11].

In this letter we study the second reference state of the open XXZ spin chain with non-diagonal boundary terms and construct the *complete* eigenstates of the model in the framework of algebraic Bethe Ansatz. In the quasi-classical limit, they give the complete eigenstates of the associated Gaudin model.

## 2. The inhomogeneous spin- $\frac{1}{2}$ XXZ open chain

Throughout,  $V$  denotes a two-dimensional linear space and  $\sigma^\pm, \sigma^z$  are the usual Pauli matrices which realize the spin- $\frac{1}{2}$  representation of the Lie algebra  $sl(2)$  on  $V$ . The spin- $\frac{1}{2}$  XXZ chain can be constructed from the well-known six-vertex model R-matrix  $\overline{R}(u) \in \text{End}(V \otimes V)$  [11] given by

$$\overline{R}(u) = \begin{pmatrix} 1 & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & 1 \end{pmatrix}. \quad (2.1)$$

The coefficient functions read:  $b(u) = \frac{\sin u}{\sin(u+\eta)}$ ,  $c(u) = \frac{\sin \eta}{\sin(u+\eta)}$ . Here we assume  $\eta$  being a generic complex number. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE),

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \quad (2.2)$$

and the unitarity, crossing-unitarity and quasi-classical properties [6]. We adopt the standard notations: for any matrix  $A \in \text{End}(V)$ ,  $A_j$  is an embedding operator in the tensor space  $V \otimes V \otimes \dots$ , which acts as  $A$  on the  $j$ -th space and as identity on the other factor spaces;  $R_{ij}(u)$  is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the  $i$ -th and  $j$ -th ones.

One introduces the ‘‘row-to-row’’ monodromy matrix  $T(u)$ , which is an  $2 \times 2$  matrix with elements being operators acting on  $V^{\otimes N}$ , where  $N = 2M$  ( $M$  being a positive integer),

$$T_0(u) = \overline{R}_{01}(u + z_1)\overline{R}_{02}(u + z_2) \cdots \overline{R}_{0N}(u + z_N). \quad (2.3)$$

Here  $\{z_j | j = 1, \dots, N\}$  are arbitrary free complex parameters which are usually called inhomogeneous parameters.

Integrable open chain can be constructed as follows [3]. Let us introduce a pair of K-matrices  $K^-(u)$  and  $K^+(u)$ . The former satisfies the reflection equation (RE)

$$\begin{aligned} \overline{R}_{12}(u_1 - u_2)K_1^-(u_1)\overline{R}_{21}(u_1 + u_2)K_2^-(u_2) = \\ K_2^-(u_2)\overline{R}_{12}(u_1 + u_2)K_1^-(u_1)\overline{R}_{21}(u_1 - u_2), \end{aligned} \quad (2.4)$$

and the latter satisfies the dual RE

$$\begin{aligned} \overline{R}_{12}(u_2 - u_1)K_1^+(u_1)\overline{R}_{21}(-u_1 - u_2 - 2\eta)K_2^+(u_2) = \\ K_2^+(u_2)\overline{R}_{12}(-u_1 - u_2 - 2\eta)K_1^+(u_1)\overline{R}_{21}(u_2 - u_1). \end{aligned} \quad (2.5)$$

For open spin-chains, instead of the standard ‘‘row-to-row’’ monodromy matrix  $T(u)$  (2.3), one needs to introduce the ‘‘double-row’’ monodromy matrix  $\mathbb{T}(u) = T(u)K^-(u)T^{-1}(-u)$ . Then the *double-row transfer matrix* is given by

$$\tau(u) = \text{tr}(K^+(u)\mathbb{T}(u)). \quad (2.6)$$

The QYBE and (dual) REs lead to that the transfer matrices with different spectral parameters commute with each other [3]:  $[\tau(u), \tau(v)] = 0$ . This ensures the integrability of the inhomogeneous spin- $\frac{1}{2}$  XXZ chain with open boundary.

In this paper, we will consider a *generic* K-matrix  $K^-(u)$  which is a generic solution to the RE (2.4) associated the six-vertex model R-matrix [12, 13]

$$K^-(u) = \begin{pmatrix} k_1^1(u) & k_2^1(u) \\ k_1^2(u) & k_2^2(u) \end{pmatrix} \equiv K(u). \quad (2.7)$$

The coefficient functions are

$$\begin{aligned} k_1^1(u) &= \frac{2 \cos(\lambda_1 - \lambda_2) - \cos(\lambda_1 + \lambda_2 + 2\xi)e^{-2iu}}{4 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}, \\ k_2^1(u) &= \frac{-i \sin(2u)e^{-i(\lambda_1 + \lambda_2)}e^{-iu}}{2 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}, \\ k_1^2(u) &= \frac{i \sin(2u)e^{i(\lambda_1 + \lambda_2)}e^{-iu}}{2 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}, \\ k_2^2(u) &= \frac{2 \cos(\lambda_1 - \lambda_2)e^{-2iu} - \cos(\lambda_1 + \lambda_2 + 2\xi)}{4 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}. \end{aligned} \quad (2.8)$$

At the same time, we introduce the corresponding *dual* K-matrix  $K^+(u)$  which is a generic solution to the dual reflection equation (2.5) with a particular choice of the free boundary parameters with respect to  $K^-(u)$ :

$$K^+(u) = \begin{pmatrix} k_1^{+1}(u) & k_2^{+1}(u) \\ k_1^{+2}(u) & k_2^{+2}(u) \end{pmatrix}. \quad (2.9)$$

The matrix elements are

$$\begin{aligned} k_1^{+1}(u) &= \frac{2 \cos(\lambda_1 - \lambda_2)e^{-i\eta} - \cos(\lambda_1 + \lambda_2 + 2\bar{\xi})e^{2iu+i\eta}}{4 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}, \\ k_2^{+1}(u) &= \frac{i \sin(2u + 2\eta)e^{-i(\lambda_1 + \lambda_2)}e^{iu-i\eta}}{2 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}, \\ k_1^{+2}(u) &= \frac{-i \sin(2u + 2\eta)e^{i(\lambda_1 + \lambda_2)}e^{iu+i\eta}}{2 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}, \\ k_2^{+2}(u) &= \frac{2 \cos(\lambda_1 - \lambda_2)e^{2iu+i\eta} - \cos(\lambda_1 + \lambda_2 + 2\bar{\xi})e^{-i\eta}}{4 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}. \end{aligned} \quad (2.10)$$

The K-matrices depend on four free boundary parameters  $\{\lambda_1, \lambda_2, \xi, \bar{\xi}\}$  which obey the constrain conditions in [4, 5, 9]. It is very convenient to introduce a vector  $\lambda = \sum_{k=1}^2 \lambda_k \epsilon_k$  associated with the boundary parameters  $\{\lambda_i\}$ , where  $\{\epsilon_i, i = 1, 2\}$  form the orthonormal basis of  $V$  such that  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ .

### 3. Vertex-face correspondence

Let us briefly review the face-type R-matrix associated with the six-vertex model. Set  $\hat{i} = \epsilon_i - \bar{\epsilon}$ ,  $\bar{\epsilon} = \frac{1}{2} \sum_{k=1}^2 \epsilon_k$ , for  $i = 1, 2$ . For a generic  $m \in V$ , define

$$m_i = \langle m, \epsilon_i \rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j \rangle, \quad i, j = 1, 2. \quad (3.1)$$

Let  $R(u, m) \in \text{End}(V \otimes V)$  be the R-matrix of the six-vertex SOS model, which is trigonometric limit of the eight-vertex SOS model [14] given by

$$R(u, m) = \sum_{i=1}^2 R_{ii}^{ii}(u, m) E_{ii} \otimes E_{ii} + \sum_{i \neq j}^2 \left\{ R_{ij}^{ij}(u, m) E_{ii} \otimes E_{jj} + R_{ij}^{ji}(u, m) E_{ji} \otimes E_{ij} \right\}, \quad (3.2)$$

where  $E_{ij}$  is the matrix with elements  $(E_{ij})_k^l = \delta_{jk} \delta_{il}$ . The coefficient functions are

$$R_{ii}^{ii}(u, \lambda) = 1, \quad (3.3)$$

$$R_{ij}^{ij}(u, \lambda) = \frac{\sin u \sin(m_{ij} - \eta)}{\sin(u + \eta) \sin(m_{ij})}, \quad i \neq j,$$

$$R_{ij}^{ji}(u, m) = \frac{\sin \eta \sin(u + m_{ij})}{\sin(u + \eta) \sin(m_{ij})}, \quad i \neq j, \quad (3.4)$$

and  $m_{ij}$  is defined in (3.1). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation (or star-triangle equation) [14].

Define the following functions:  $\theta^{(1)}(u) = e^{-iu}$ ,  $\theta^{(2)}(u) = 1$ . Let us introduce two intertwiners which are 2-component column vectors  $\phi_{m, m-\eta \hat{j}}(u)$  labelled by  $\hat{1}, \hat{2}$ . The  $k$ -th element of  $\phi_{m, m-\eta \hat{j}}(u)$  is given by

$$\phi_{m, m-\eta \hat{j}}^{(k)}(u) = \theta^{(k)}(u + 2m_j). \quad (3.5)$$

Explicitly,

$$\phi_{m, m-\eta \hat{1}}(u) = \begin{pmatrix} e^{-i(u+2m_1)} \\ 1 \end{pmatrix}, \quad \phi_{m, m-\eta \hat{2}}(u) = \begin{pmatrix} e^{-i(u+2m_2)} \\ 1 \end{pmatrix}. \quad (3.6)$$

Obviously, the two intertwiner vectors  $\phi_{m, m-\eta \hat{i}}(u)$  are linearly *independent* for a generic  $m \in V$ .

Using the intertwiner vectors, one can derive the following face-vertex correspondence relation [5]

$$\begin{aligned} \bar{R}_{12}(u_1 - u_2) \phi_{m, m-\eta \hat{i}}(u_1) \otimes \phi_{m-\eta \hat{i}, m-\eta(\hat{i}+\hat{j})}(u_2) = \\ \sum_{k, l} R(u_1 - u_2, m)_{ij}^{kl} \phi_{m-\eta \hat{l}, m-\eta(\hat{l}+\hat{k})}(u_1) \otimes \phi_{m, m-\eta \hat{l}}(u_2). \end{aligned} \quad (3.7)$$

Then the QYBE (2.2) of the vertex-type R-matrix  $\bar{R}(u)$  is equivalent to the dynamical Yang-Baxter equation of the SOS R-matrix  $R(u, m)$ . For a generic  $m$ , we can introduce other types of intertwiners  $\bar{\phi}, \tilde{\phi}$  satisfying the conditions,

$$\sum_{k=1}^2 \bar{\phi}_{m, m-\eta \hat{\mu}}^{(k)}(u) \phi_{m, m-\eta \hat{\nu}}^{(k)}(u) = \delta_{\mu\nu}, \quad \sum_{k=1}^2 \tilde{\phi}_{m+\eta \hat{\mu}, m}^{(k)}(u) \phi_{m+\eta \hat{\nu}, m}^{(k)}(u) = \delta_{\mu\nu}. \quad (3.8)$$

One may verify that the K-matrices  $K^\pm(u)$  given by (2.7) and (2.9) can be expressed in terms of the intertwiners and *diagonal* matrices  $\mathcal{K}(\lambda|u)$  and  $\tilde{\mathcal{K}}(\lambda|u)$  as follows

$$K^-(u)_t^s = \sum_{i, j} \phi_{\lambda-\eta(\hat{i}-\hat{j}), \lambda-\eta \hat{i}}^{(s)}(u) \mathcal{K}(\lambda|u)_i^j \bar{\phi}_{\lambda, \lambda-\eta \hat{i}}^{(t)}(-u), \quad (3.9)$$

$$K^+(u)_t^s = \sum_{i, j} \phi_{\lambda, \lambda-\eta \hat{j}}^{(s)}(-u) \tilde{\mathcal{K}}(\lambda|u)_i^j \tilde{\phi}_{\lambda-\eta(\hat{j}-\hat{i}), \lambda-\eta \hat{j}}^{(t)}(u). \quad (3.10)$$

Here the two *diagonal* matrices  $\mathcal{K}(\lambda|u)$  and  $\tilde{\mathcal{K}}(\lambda|u)$  are given by

$$\begin{aligned} \mathcal{K}(\lambda|u) &\equiv \text{Diag} (k(\lambda|u)_1, k(\lambda|u)_2) \\ &= \text{Diag} \left( \frac{\sin(\lambda_1 + \xi - u)}{\sin(\lambda_1 + \xi + u)}, \frac{\sin(\lambda_2 + \xi - u)}{\sin(\lambda_2 + \xi + u)} \right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \tilde{\mathcal{K}}(\lambda|u) &\equiv \text{Diag} (\tilde{k}(\lambda|u)_1, \tilde{k}(\lambda|u)_2) \\ &= \text{Diag} \left( \frac{\sin(\lambda_{12} - \eta) \sin(\lambda_1 + \xi + u + \eta)}{\sin \lambda_{12} \sin(\lambda_1 + \xi - u - \eta)}, \frac{\sin(\lambda_{12} + \eta) \sin(\lambda_2 + \xi + u + \eta)}{\sin \lambda_{12} \sin(\lambda_2 + \xi - u - \eta)} \right). \end{aligned} \quad (3.12)$$

Although the K-matrices  $K^\pm(u)$  given by (2.7) and (2.9) are generally non-diagonal (in the vertex form), after the face-vertex transformations (3.9) and (3.10), the face type counterparts  $\mathcal{K}(\lambda|u)$  and  $\tilde{\mathcal{K}}(\lambda|u)$  *simultaneously* become diagonal. This fact enables us to apply the generalized algebraic Bethe ansatz method developed in [15] for SOS type integrable models to diagonalize the transfer matrices  $\tau(u)$  (2.6).

The decomposition of  $K^+(u)$  (3.10) and the diagonal property (3.12) lead to the recasting of the transfer matrix  $\tau(u)$  (2.6) in the following face type form

$$\tau(u) = \text{tr}(K^+(u)\mathbb{T}(u)) = \sum_{\mu, \nu} \tilde{\mathcal{K}}(\lambda|u)_\nu^\mu \mathcal{T}(\lambda|u)_\mu^\nu = \sum_{\mu} \tilde{k}(\lambda|u)_\mu \mathcal{T}(\lambda|u)_\mu^\mu. \quad (3.13)$$

Here we have introduced the face-type double-row monodromy matrix  $\mathcal{T}(m|u)$

$$\mathcal{T}(m|u)_\mu^\nu = \sum_{i,j} \tilde{\phi}_{m-\eta(\hat{\mu}-\hat{\nu}), m-\eta\hat{\mu}}^{(j)}(u) \mathbb{T}(u)_i^j \phi_{m, m-\eta\hat{\mu}}^{(i)}(-u). \quad (3.14)$$

This face-type double-row monodromy matrix can be expressed in terms of the face type R-matrix  $R(m|u)$  (3.2) and K-matrix  $\mathcal{K}(\lambda|u)$  (3.11) (for the details, see equation (4.19) of [15]).

As in [6], let us introduce operators:

$$\mathcal{A}(m|u) \equiv \mathcal{A}^{(1)}(m|u) = \mathcal{T}(m|u)_1^1, \quad \mathcal{B}(m|u) = \frac{\mathcal{T}(m|u)_2^1}{\sin(m_{12})}, \quad \mathcal{C}(m|u) = \frac{\mathcal{T}(m|u)_1^2}{\sin(m_{21})}, \quad (3.15)$$

$$\mathcal{D}(m|u) \equiv \mathcal{D}^{(1)}(m|u) = \frac{\sin(m_{12} + \eta)}{\sin(m_{12})} \left\{ \mathcal{T}(m|u)_2^2 - R(2u, m + \eta\hat{1})_{12}^2 \mathcal{A}^{(1)}(m|u) \right\}. \quad (3.16)$$

We remark that the transfer matrix  $\tau(u)$  (2.6) can be expressed in terms of the operators  $\mathcal{A}^{(1)}$  and  $\mathcal{D}^{(1)}$ . It was found in [6] that one can construct a reference state, denoted by  $|\Omega^{(1)}(\lambda)\rangle$ ,

$$|\Omega^{(1)}(\lambda)\rangle = \phi_{\lambda-(N-1)\eta\hat{1}, \lambda-N\eta\hat{1}}(-z_1) \otimes \phi_{\lambda-(N-2)\eta\hat{1}, \lambda-(N-1)\eta\hat{1}}(-z_2) \cdots \otimes \phi_{\lambda, \lambda-\eta\hat{1}}(-z_N), \quad (3.17)$$

in the sense that the state is common eigenstate of the operators  $\mathcal{A}^{(1)}$  and  $\mathcal{D}^{(1)}$  and is annihilated by  $\mathcal{C}$  (c.f. (4.6)). The associated Bethe states can be constructed by applying the ‘‘creation operator’’  $\mathcal{B}$  on the corresponding reference state

$$|v_1, \dots, v_M\rangle^{(1)} = \mathcal{B}(\lambda - 2\eta\hat{1}|v_1) \mathcal{B}(\lambda - 4\eta\hat{1}|v_2) \cdots \mathcal{B}(\lambda - 2M\eta\hat{1}|v_M) |\Omega^{(1)}(\lambda)\rangle. \quad (3.18)$$

If the parameters  $\{v_k\}$  satisfy the following Bethe Ansatz equations,

$$\begin{aligned} & \frac{\sin(\lambda_2 + \xi + v_\alpha) \sin(\lambda_2 + \bar{\xi} - v_\alpha) \sin(\lambda_1 + \bar{\xi} + v_\alpha) \sin(\lambda_1 + \xi - v_\alpha)}{\sin(\lambda_2 + \bar{\xi} + v_\alpha + \eta) \sin(\lambda_2 + \xi - v_\alpha - \eta) \sin(\lambda_1 + \xi + v_\alpha + \eta) \sin(\lambda_1 + \bar{\xi} - v_\alpha - \eta)} \\ &= \prod_{k \neq \alpha}^M \frac{\sin(v_\alpha + v_k + 2\eta) \sin(v_\alpha - v_k + \eta)}{\sin(v_\alpha + v_k) \sin(v_\alpha - v_k - \eta)} \\ & \quad \times \prod_{k=1}^{2M} \frac{\sin(v_\alpha + z_k) \sin(v_\alpha - z_k)}{\sin(v_\alpha + z_k + \eta) \sin(v_\alpha - z_k + \eta)}, \quad \alpha = 1, \dots, M, \end{aligned} \quad (3.19)$$

the Bethe state  $|v_1, \dots, v_M\rangle^{(1)}$  becomes the eigenstate of the transfer matrix with eigenvalue  $\Lambda^{(1)}(u)$  given by [6]

$$\begin{aligned} \Lambda^{(1)}(u) &= \frac{\sin(\lambda_2 + \bar{\xi} - u) \sin(\lambda_1 + \bar{\xi} + u) \sin(\lambda_1 + \xi - u) \sin(2u + 2\eta)}{\sin(\lambda_2 + \bar{\xi} - u - \eta) \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_1 + \xi + u) \sin(2u + \eta)} \\ & \quad \times \prod_{k=1}^M \frac{\sin(u + v_k) \sin(u - v_k - \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)} \\ & \quad + \frac{\sin(\lambda_2 + \bar{\xi} + u + \eta) \sin(\lambda_1 + \xi + u + \eta) \sin(\lambda_2 + \xi - u - \eta) \sin 2u}{\sin(\lambda_2 + \bar{\xi} - u - \eta) \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u) \sin(2u + \eta)} \\ & \quad \times \prod_{k=1}^M \frac{\sin(u + v_k + 2\eta) \sin(u - v_k + \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)} \prod_{k=1}^{2M} \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)}. \end{aligned} \quad (3.20)$$

#### 4. Second reference state and associated Bethe states

Let us introduce the second reference state  $|\Omega^{(2)}(\lambda)\rangle$ ,

$$|\Omega^{(2)}(\lambda)\rangle = \phi_{\lambda - (N-1)\eta\hat{2}, \lambda - N\eta\hat{2}}(-z_1) \otimes \phi_{\lambda - (N-2)\eta\hat{2}, \lambda - (N-1)\eta\hat{2}}(-z_2) \cdots \otimes \phi_{\lambda, \lambda - \eta\hat{2}}(-z_N), \quad (4.1)$$

and the associated operators  $\mathcal{A}^{(2)}$  and  $\mathcal{D}^{(2)}$  which are linear combinations of  $\{\mathcal{T}(m|u)_i^i\}$ ,

$$\mathcal{A}^{(2)}(m|u) = \frac{\sin(m_{21} + \eta)}{\sin(m_{21})} \{ \mathcal{T}(m|u)_1^1 - R(2u, m + \eta\hat{2})_{21}^{12} \mathcal{D}^{(2)}(m|u) \}, \quad (4.2)$$

$$\mathcal{D}^{(2)}(m|u) = \mathcal{T}(m|u)_2^2. \quad (4.3)$$

Using the technique developed in [15], after tedious calculations, we find that the state  $|\Omega^{(2)}(\lambda)\rangle$  given by (4.1) is *exactly* the reference state in the following sense,

$$\begin{aligned} \mathcal{A}^{(2)}(\lambda - N\eta\hat{2}|u) |\Omega^{(2)}(\lambda)\rangle &= \frac{\sin 2u \sin(\lambda_2 + \xi + u + \eta) \sin(\lambda_1 + \xi - u - \eta)}{\sin(2u + \eta) \sin(\lambda_2 + \xi + u) \sin(\lambda_1 + \xi + u)} \\ & \quad \times \left\{ \prod_{k=1}^N \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)} \right\} |\Omega^{(2)}(\lambda)\rangle, \end{aligned} \quad (4.4)$$

$$\mathcal{D}^{(2)}(\lambda - N\eta\hat{2}|u) |\Omega^{(2)}(\lambda)\rangle = \frac{\sin(\lambda_2 + \xi - u)}{\sin(\lambda_2 + \xi + u)} |\Omega^{(2)}(\lambda)\rangle, \quad (4.5)$$

$$\mathcal{B}(\lambda - N\eta\hat{2}|u) |\Omega^{(2)}(\lambda)\rangle = 0, \quad (4.6)$$

$$\mathcal{C}(\lambda - N\eta\hat{2}|u) |\Omega^{(2)}(\lambda)\rangle \neq 0. \quad (4.7)$$

Then the second set of Bethe states can be constructed by applying the “creation operator”  $\mathcal{C}$  on the reference state  $|\Omega^{(2)}(\lambda)\rangle$  (c.f. (3.18))

$$|v_1, \dots, v_M\rangle^{(2)} = \mathcal{C}(\lambda - 2\eta\hat{2}|v_1)\mathcal{C}(\lambda - 4\eta\hat{2}|v_2)\cdots\mathcal{C}(\lambda - 2M\eta\hat{2}|v_M)|\Omega^{(2)}(\lambda)\rangle. \quad (4.8)$$

One may check that the transfer matrix  $\tau(u)$  (2.6) is a linear combination of the operators  $\mathcal{A}^{(2)}$  and  $\mathcal{D}^{(2)}$

$$\begin{aligned} \tau(u) &= \frac{\sin(\lambda_1 + \bar{\xi} + u + \eta)}{\sin(\lambda_1 + \bar{\xi} - u - \eta)} \mathcal{A}^{(2)}(\lambda|u) \\ &\quad + \frac{\sin(\lambda_1 + \bar{\xi} - u) \sin(\lambda_2 + \bar{\xi} + u) \sin(2u + 2\eta)}{\sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta) \sin(2u + \eta)} \mathcal{D}^{(2)}(\lambda|u). \end{aligned} \quad (4.9)$$

Carrying out the generalized Bethe Ansatz [6, 15], we finally find that if the parameters  $\{v_k\}$  satisfy the second Bethe Ansatz equations (comparing with the first ones (3.19)),

$$\begin{aligned} &\frac{\sin(\lambda_1 + \xi + v_\alpha) \sin(\lambda_1 + \bar{\xi} - v_\alpha) \sin(\lambda_2 + \bar{\xi} + v_\alpha) \sin(\lambda_2 + \xi - v_\alpha)}{\sin(\lambda_1 + \bar{\xi} + v_\alpha + \eta) \sin(\lambda_1 + \bar{\xi} - v_\alpha - \eta) \sin(\lambda_2 + \bar{\xi} + v_\alpha + \eta) \sin(\lambda_2 + \bar{\xi} - v_\alpha - \eta)} \\ &= \prod_{k \neq \alpha}^M \frac{\sin(v_\alpha + v_k + 2\eta) \sin(v_\alpha - v_k + \eta)}{\sin(v_\alpha + v_k) \sin(v_\alpha - v_k - \eta)} \\ &\quad \times \prod_{k=1}^{2M} \frac{\sin(v_\alpha + z_k) \sin(v_\alpha - z_k)}{\sin(v_\alpha + z_k + \eta) \sin(v_\alpha - z_k + \eta)}, \quad \alpha = 1, \dots, M, \end{aligned} \quad (4.10)$$

the Bethe states  $|v_1, \dots, v_M\rangle^{(2)}$  yield the second set of the eigenstates of the transfer matrix with the eigenvalues,

$$\begin{aligned} \Lambda^{(2)}(u) &= \frac{\sin(2u + 2\eta) \sin(\lambda_1 + \bar{\xi} - u) \sin(\lambda_2 + \bar{\xi} + u) \sin(\lambda_2 + \xi - u)}{\sin(2u + \eta) \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \xi + u)} \\ &\quad \times \prod_{k=1}^M \frac{\sin(u + v_k) \sin(u - v_k - \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)} \\ &\quad + \frac{\sin(2u) \sin(\lambda_1 + \bar{\xi} + u + \eta) \sin(\lambda_2 + \xi + u + \eta) \sin(\lambda_1 + \xi - u - \eta)}{\sin(2u + \eta) \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \xi + u) \sin(\lambda_1 + \xi + u)} \\ &\quad \times \prod_{k=1}^M \frac{\sin(u + v_k + 2\eta) \sin(u - v_k + \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)} \prod_{k=1}^{2M} \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)}. \end{aligned} \quad (4.11)$$

Note that the normalizations adopted in this paper for the R- and K-matrices are different from those used in [9]. After rescaling an overall factor  $\sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta) \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u) \prod_{k=1}^N \sin(u + z_k + \eta) \sin(u - z_k + \eta)$  and setting all inhomogeneous parameters  $z_k = 0$ , our two eigenvalues  $\{\Lambda^{(i)}(u)\}$  recover those in [9]. Therefore two sets Bethe states  $\{|v_1, \dots, v_M\rangle^{(i)}\}$  (3.18) and (4.8) together constitute the complete eigenstates of the transfer matrix  $\tau(u)$  (2.6).

## 5. Results for the Gaudin model

In order to study the associated Gaudin model, we need further restrict  $\bar{\xi} = \xi$  [6]. Following [16, 17] one can introduce the corresponding Gaudin operators  $\{H_j\}$  by expanding the



double-row transfer matrix  $\tau(u)$  (2.6) at the point  $u = z_j$  around  $\eta = 0$ :

$$\tau(z_j) = \text{id} + \eta H_j + O(\eta^2), \quad \text{with } H_j = \frac{\partial}{\partial \eta} \tau(z_j)|_{\eta=0}, \quad j = 1, \dots, N, \quad (5.1)$$

where

$$\begin{aligned} H_j = & \Gamma_j(z_j) + \sum_{k \neq j}^{2M} \frac{1}{\sin(z_j - z_k)} \left\{ \sigma_k^+ \sigma_j^- + \sigma_k^- \sigma_j^+ + \cos(z_j - z_k) \frac{\sigma_k^z \sigma_j^z - 1}{2} \right\} \\ & + \sum_{k \neq j}^{2M} \frac{K_j^{-1}(z_j)}{\sin(z_j + z_k)} \left\{ \sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+ + \cos(z_j + z_k) \frac{\sigma_j^z \sigma_k^z - 1}{2} \right\} K_j(z_j), \end{aligned} \quad (5.2)$$

where

$$\Gamma_j(u) = \frac{\partial}{\partial \eta} \{ \bar{K}_j(u) \}|_{\eta=0} K_j(u) \quad j = 1, \dots, N,$$

with

$$\bar{K}_j(u) = \text{tr}_0 \{ K_0^+(u) \bar{R}_{0j}(2u) R_{0j}(0) \}.$$

The commutativity of the transfer matrices  $\{\tau(z_j)\}$  for a generic  $\eta$  implies  $[H_j, H_k] = 0$ , for  $i, j = 1, \dots, N$ . Thus the Gaudin system defined by (5.2) is integrable. Moreover the relation (5.1) between  $\{H_j\}$  and  $\{\tau(z_j)\}$  enable us to extract the eigenstates of the Gaudin operators and the corresponding eigenvalues from the results obtained in last section.

Let us introduce states  $|\bar{\Omega}^{(i)}(\lambda)\rangle$ ,

$$|\bar{\Omega}^{(i)}(\lambda)\rangle = \begin{pmatrix} e^{i(z_1 - 2\lambda_i)} \\ 1 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} e^{i(z_N - 2\lambda_i)} \\ 1 \end{pmatrix}, \quad i = 1, 2. \quad (5.3)$$

These states can be obtained from the reference states  $|\Omega^{(i)}(\lambda)\rangle$  by taking the limit:  $|\bar{\Omega}^{(i)}(\lambda)\rangle = \lim_{\eta \rightarrow 0} |\Omega^{(i)}(\lambda)\rangle$ . Let us introduce a matrix  $C(u) \in \text{End}(V)$  associated with the intertwiner vector  $\phi$  (3.5)

$$C(u) = \begin{pmatrix} e^{-i(u+2\lambda_1)} & e^{-i(u+2\lambda_2)} \\ 1 & 1 \end{pmatrix}, \quad (5.4)$$

and the corresponding gauged Pauli operator  $\sigma^\pm(u) \in \text{End}(V)$ :  $\sigma^\pm(u) = C(u)\sigma^\pm C(u)^{-1}$ . Then we can construct states  $\Psi^{(i)}(x_1, \dots, x_M)$ :

$$\begin{aligned} \Psi^{(1)}(x_1, \dots, x_M) = & \prod_{\alpha=1}^M \left( \sum_{k=1}^{2M} \left\{ \frac{\sin(\lambda_1 + \xi - x_\alpha) \sin(x_\alpha - z_k + \lambda_{12})}{\sin(\lambda_1 + \xi + x_\alpha) \sin(x_\alpha - z_k)} \right. \right. \\ & \left. \left. - \frac{\sin(\lambda_2 + \xi - x_\alpha) \sin(x_\alpha + z_k - \lambda_{12})}{\sin(\lambda_2 + \xi + x_\alpha) \sin(x_\alpha + z_k)} \right\} \sigma_k^-(-z_k) \right) |\bar{\Omega}^{(1)}(\lambda)\rangle, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \Psi^{(2)}(x_1, \dots, x_M) = & \prod_{\alpha=1}^M \left( \sum_{k=1}^{2M} \left\{ \frac{\sin(\lambda_2 + \xi - x_\alpha) \sin(x_\alpha - z_k - \lambda_{12})}{\sin(\lambda_2 + \xi + x_\alpha) \sin(x_\alpha - z_k)} \right. \right. \\ & \left. \left. - \frac{\sin(\lambda_1 + \xi - x_\alpha) \sin(x_\alpha + z_k + \lambda_{12})}{\sin(\lambda_1 + \xi + x_\alpha) \sin(x_\alpha + z_k)} \right\} \sigma_k^+(-z_k) \right) |\bar{\Omega}^{(2)}(\lambda)\rangle. \end{aligned} \quad (5.6)$$

Noting the relations (5.1) and using the same method as in [6], we find that if the parameters  $\{x_k\}$  satisfy the following Bethe Ansatz equations

$$\sum_{j=1}^2 \frac{1}{\sin(\lambda_j + \xi - x_\alpha) \sin(\lambda_j + \xi + x_\alpha)} + \sum_{k=1}^{2M} \frac{1}{\sin(x_\alpha + z_k) \sin(x_\alpha - z_k)} = 2 \sum_{k \neq \alpha}^M \frac{1}{\sin(x_\alpha + x_k) \sin(x_\alpha - x_k)}, \quad \alpha = 1, \dots, M, \quad (5.7)$$

the two sets of states  $\Psi^{(i)}(x_1, \dots, x_M)$  constitute the *entire* eigenstates of the Gaudin operators

$$H_j \Psi^{(i)}(x_1, \dots, x_M) = E_j \Psi^{(i)}(x_1, \dots, x_M), \quad i = 1, 2. \quad (5.8)$$

The functions  $E_j$  is

$$E_j = \cot 2z_j + \sum_{j=1}^2 \cot(\lambda_j + \xi - z_j) + \sum_{k=1}^M \frac{\sin 2z_j}{\sin(x_k - z_j) \sin(x_k + z_j)}. \quad (5.9)$$

## 6. Conclusions

We have studied the second reference state of the open XXZ spin chain with non-diagonal boundary term, which leads to the second set of Bethe state (4.8). These Bethe states give rise to the corresponding Bethe Ansatz equations (4.10) and eigenvalues (4.11) proposed in [10, 9] by the functional Bethe Ansatz method. In the quasi-classical limit, two sets of Bethe states (5.5) and (5.6) constitute the complete eigenstates of the associated Gaudin model.

Very recently, an exact solution of the eigenvalue of the transfer matrix of open XXZ spin chain for arbitrary boundary parameters was proposed by functional Bethe Ansatz [18] and by representation of q-Onsager algebra [19]. It would be interesting to rederive their results in the framework of algebraic Bethe Ansatz. Moreover, such structure of multiply reference states found here also appears in open spin chains associated with higher rank algebras [20].

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